Lower Bounds for Testing Digraph Connectivity with One-Pass Streaming Algorithms

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Abstract—In this note, we show that three graph properties—strong connectivity, acyclicity, and reachability from a vertex s to all vertices—each require a working memory of $\Omega(\epsilon m)$ on a graph with m edges to be determined correctly with probability greater than $(1+\epsilon)/2$.

Index Terms—Lower bounds, streaming graph problems, strong connectivity, acyclicity

1 Introduction

In the streaming model of computation, the input is given as a sequence, or *stream*, of elements. There is no random access to the elements; the sequence must be scanned in order. The goal is to process the stream using a small amount of working memory. For an overview see [8]. There has been much research devoted to the study of streaming algorithms, most notably the Gödel-prize winning work of Alon, Matias, and Szegedy, on the space complexity of randomized algorithms to approximate frequency moments of a sequence when the sequence is given one element at a time and cannot be stored [2].

2 Related Work

For undirected graph problems, there are many lower bounds in the edge streaming model. Sun and Woodruff provide a tight $\Omega(n \log n)$ lower bound for the following graph problems: testing bipartiteness, connectivity, cycle-freeness, whether a graph is Eulerian, planarity, H-minor freeness, finding a minimum spanning tree of a connected graph, and testing if the diameter of a sparse graph is constant. They also give an $\Omega(nk \log n)$ lower bound for: deterministic algorithms for k-edge connectivity and k-vertex connectivity [9].

Guha, McGregor, and Tench give an algorithm for k-vertex connectivity that takes $O(kn \text{ polylog } n)$ space [5].

Feigenbaum, Kannan, and Zhang show that any exact, deterministic algorithm for computing the diameter of an undirected graph in the Euclidean plane requires $\Omega(n)$ bits of working memory [4]. Zelke shows that any algorithm that is able to find a minimum cut of an undirected graph requires $\Omega(m)$ bits of working memory, this remains true even if randomization is allowed [10].

For directed graphs problems, the ones most likely to come up in analyses of the internet, much less is known. Henzinger et al. showed that for any $0 < \epsilon < 1$, estimating the size of the transitive closure of a DAG with relative expected error ϵ requires $\Omega(m)$ bits of working memory [7]. Feigenbaum, Kannan, McGregor, Suri, and Zhang showed that testing reachability from a given vertex s to another given vertex t requires $\Omega(m)$ bits of space [3]. Guruswami and Onak showed that even with p passes, the problem requires $\Omega(n^{1+1/(2(p+1))}/p^{20} \log^{3/2} n)$ bits of space to be solvable with probability at least 9/10 [6].

As for upper bounds in undirected graphs, there are one-pass algorithms for connected components, k-edge and k-vertex connectivity ($k \leq 3$), and planarity testing that use $O(n \log n)$ bits of working memory [7]. There is an algorithm that approximates the diameter within $1+\epsilon$ using $O(\frac{1}{\epsilon})$ bits [4]. For upper bounds in directed graphs, there is an algorithm that computes the exact size of the transitive closure using $O(m \log n)$ bits of working memory [7].

3 Lower Bounds

In this short note, we consider three basic connectivity questions in directed graphs: determining if a graph is strongly connected, determining if a graph is acyclic, and determining if a vertex s reaches all other vertices. A directed graph $G = (V, E)$ is said to be *strongly connected* if for every pair of vertices $u, v \in V$ there is a path from u to v and a path from v to u. A directed graph $G = (V, E)$ is said to be *acyclic* if G contains no cycles. We say that a vertex s *reaches* a vertex v if there is a directed path from s to v.

We show that, even with randomization, these graph properties each require $\Omega(m)$ bits of working memory to be decided with probability greater than $(1+\epsilon)/2$ by a one-pass streaming algorithm on n vertices and m edges. For these lower bounds we will use simple reductions from the index problem (or the bit-vector problem) in communication complexity:

> *Alice has a bit-vector x of length m. Bob has an index $i \in \{1, 2, \ldots, m\}$ and wishes to know the ith bit of x. The only communication allowed is from Alice to Bob.*

The following is a rewording of Theorem 2 from Ablayev [1].

Theorem 1. *For Bob to correctly determine x_i with probability better than $\frac{(1+\epsilon)}{2}$, $\Omega(\epsilon m)$ bits of communication are required.*

We will now state and prove our main Lemma:

Lemma 2. *Any algorithm that correctly determines the following graph properties with probability better than $\frac{(1+\epsilon)}{2}$ requires $\Omega(\epsilon m)$ bits of working memory:*

- *acyclicity*
- *strong connectivity*
- *reachability of all from s.*

Proof. We reduce from the index problem and use Theorem 1. Let x denote the m-bit vector owned by Alice. We define the stream using two sets of edges E_1, E_2. The edge stream first has the edges of E_1 in arbitrary order, followed by the edges of E_2, also in arbitrary order. The set E_1 is entirely determined by the m-bit vector x owned by Alice, and the set E_2 is entirely determined by the index i owned by Bob. The graph defined by $E_1 \cup E_2$ has $\Omega(\sqrt{m})$ vertices, and E_1 has $O(m)$ edges. To solve the index problem, Alice constructs E_1 and simulates the streaming algorithm up to the point when E_1 has arrived, then sends to Bob the current state of the memory. Upon reception of the message, Bob constructs E_2 and continues the simulation up to the point when E_2 has finished arriving. Bob's final decision is then determined by the outcome of the streaming algorithm. Thus, the lemma will be proved.

Acyclicity. Let $n = \lceil \sqrt{m} \rceil$ and let $V = L \cup R$, where L and R both have size n and have vertices labeled 0 through $n-1$. Let E_1 be the bipartite graph that has an edge from vertex $\ell_j \in L$ to vertex $r_k \in R$ iff x has a 1 in position $jn + k$. Let E_2 consist of a single edge determined by Bob's bit i: let $k = i \bmod n$ and $j = \frac{i-k}{n}$. Then E_2 consists of the edge from vertex $r_k \in R$ to vertex $\ell_j \in L$. See Figs. 1a and 1b for an illustration of an example E_1 and E_2.

Observe that $E_1 \cup E_2$ is acyclic iff $x_i = 0$, thus the reduction is complete.

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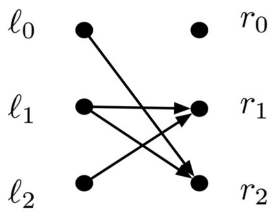
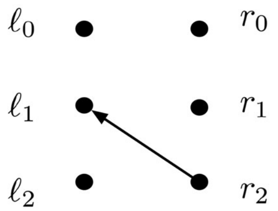


(a) An example of E_1 corresponding to the bit-vector 001011010.

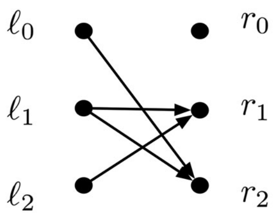


(b) An example of E_2 corresponding to B's index being 5, $j = 1$ and $k = 2$.

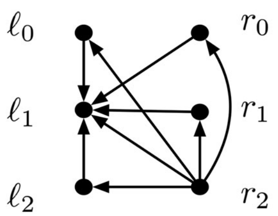
Fig. 1. An example acyclicity reduction.

Strong Connectivity. The construction for E_1 is the same as in the acyclic case. Let E_2 consist of $4n - 2$ edges determined by Bob's bit i: let $k = i \bmod n$ and $j = \frac{i-k}{n}$. Then E_2 consists of all edges from $r_k \in R$ to each vertex $V - \{r_k\}$, and from $V - \{\ell_j\}$ to $\ell_j \in L$. See Figs. 2a and 2b for an illustration of an example E_1 and E_2.

We claim that G is strongly connected iff $x_i = 1$. Indeed, if G is strongly connected, there must be a path from ℓ_j to r_k. The only edges leaving ℓ_j are to vertices in R and the only edges entering r_k are from vertices in L. And the only edges extending from R to L are either entering ℓ_j or leaving r_k. Thus, the only possible path from ℓ_j to r_k is the single edge from ℓ_j to r_k which is present only when $x_i = 1$. Now suppose that $x_i = 1$. k can certainly reach every vertex and every vertex can reach ℓ_j. Since the edge from ℓ_j to r_k is present, we know that every vertex can reach r_k and r_k can reach every vertex. Therefore, G is strongly connected. Thus the reduction is complete.

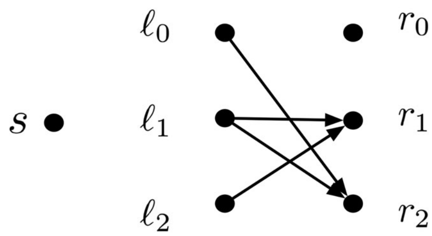


(a) An example of E_1 corresponding to the bit-vector 001011010.

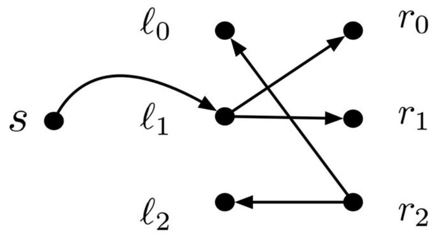


(b) An example of E_2 corresponding to B's index being 5, $j = 1$ and $k = 2$.

Fig. 2. An example strong connectivity reduction.



(a) An example of E_1 corresponding to the bit-vector 001011010.



(b) An example of E_2 corresponding to B's index being 5, $j = 1$ and $k = 2$.

Fig. 3. An example reachability reduction.

Reachability from s. Let E_1 be as above with additional vertex s with in and out degree 0.

Let E_2 consist of $2n - 1$ edges determined by Bob's bit i: let $k = i \bmod n$ and $j = \frac{i-k}{n}$. Then E_2 consists of one edge from s to $\ell_j \in L$, $n - 1$ edges from $\ell_j \in L$ to $R - \{r_k\}$, and $n - 1$ edges from $r_k \in R$ to $L - \{\ell_j\}$. See Figs. 3a and 3b for an illustration of an example E_1 and E_2.

We claim that in G s reaches everything iff $x_i = 1$. Indeed, if s can reach all vertices in G, and the only edge from s is to ℓ_j, ℓ_j must be able to reach all vertices in $G - \{s\}$. In particular ℓ_j must reach r_k. The only edges extending from R to L are from r_k, so the only way for ℓ_j to reach r_k is by the edge from ℓ_j to r_k which is present only when $x_i = 1$. Now suppose $x_i = 1$. We know s reaches ℓ_j and therefore all of R, including r_k, and r_k reaches all of $L - \{j\}$. Therefore, s reaches all vertices of G. Thus the reduction is complete. □

4 Conclusion

We have shown that any one-pass algorithm that correctly determines the acyclicity, strong connectivity, or reachability of a graph with probability better than $\frac{(1+\epsilon)}{2}$ requires $\Omega(\epsilon m)$ bits of working memory. Future work includes determining if the same result holds for an multiple-pass streaming algorithm and determining if any other graph properties exhibit a similar bound.

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